TOPOLOGICAL DEGREE THEORY

(Nirenberg, Ambrosetti – Malchiodi)

Topological method to solve the eq $f(x)=y$ in a bd domain

fin dim $\rightsquigarrow$ Brouwer

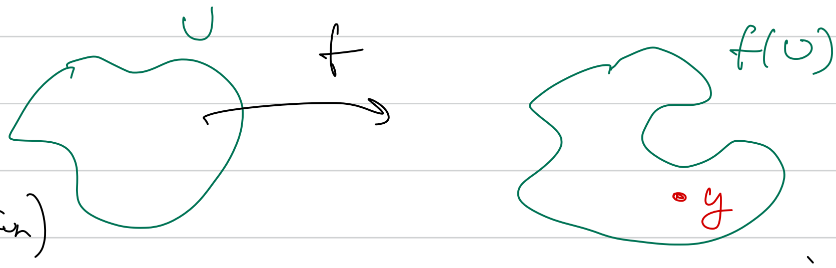
inf dim $\rightsquigarrow$ Schauder

Topological: stable under homotopy

SETUP Let $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^n$, open - bd, $\mathcal{V}$ connected and $y \in \mathcal{V}$

$$D_y(\mathcal{U}; \mathcal{V}) = \{ f : \overline{\mathcal{U}} \to \mathcal{V},\ \text{continuous},\ y \notin f(\partial \mathcal{U}) \}$$

($y \notin f(\partial U)$ because we want stability under perturbation)



Def Let ω be n-form, with compact support and

i) $y \in \operatorname{supp} \omega \subset \mathcal{V} \setminus f(\partial \mathcal{U})$

ii) $\int_{\mathcal{V}} \omega = 1$

admissible

the DEGREE of $f \in D_y(\mathcal{U}; \mathcal{V}) \cap C^1$ at y is

$\omega = \omega(x)\, dx_1 \wedge \dots \wedge dx_n$

$$\deg(f, \mathcal{U}, y) := \int_U f^* \omega = \int_U \omega(f(x)) \det df(x)\, dx_1 \dots dx_n$$

Obs The degree is well defined: Let ω_1 another admissible n-form

then $\int_V \omega - \omega_1 = 0 \iff \omega - \omega_1 = d\mu$

$\operatorname{supp} \mu \subseteq \mathcal{V} \setminus f(\partial \mathcal{U})$

in view of

POINCARÉ LEMMA Let $\Omega \subset \mathbb{R}^n$ connected, open, ω n-form with compact support $\mathrm{supp}(\omega) \subset \Omega$

Then $\int_V \omega = 0 \iff \omega = d\mu$, μ $(n-1)$ form, $\mathrm{supp}\,\mu \subset \Omega$

$$\rightsquigarrow \int_U f^* \omega - \int_U f^* \omega_1 = \int_U f^* (\omega - \omega_1) = \int_U f^* d\mu$$
$$= \int_U d\, f^* \mu = \int_{\partial U} f^* \mu = 0$$

PROPERTIES OF DEGREE I

(P1) NORMALIZATION: $\deg(\mathbb{1}_{\mathbb{R}^n}, \mathcal{U}, y) = \begin{cases} 1 & y \in \mathcal{U} \\ 0 & y \notin \mathcal{U} \end{cases}$

(P2) SOLUTION: $\deg(f, \mathcal{U}, y) \neq 0 \Rightarrow \exists z \in \mathcal{U} : f(z) = y$

(P3) TRANSLATION: $\deg(f, \mathcal{U}, y) = \deg(f - y, \mathcal{U}, 0)$

(P4) DECOMPOSITION: if $\Omega_1 \cap \Omega_2 = \emptyset$
$\deg(f, \Omega_1 \cup \Omega_2, y) = \deg(f, \Omega_1, y) + \deg(f, \Omega_2, y)$

(P5) HOMOTOPY: if $h(t,x)$ is admissible homotopy, i.e.

- $h \in C([0,1] \times \overline{\mathcal{U}}; \mathbb{R}^n)$
- $h(t,x) \neq y \quad \forall (t,x) \in [0,1] \times \partial \Omega$

then $\deg(h(t,\cdot), \mathcal{U}, y)$ is constant in t

(P6) LOCAL CONSTANT: For $g \in D_y(\mathcal{U}, V)$ e $\|g - f\|_{e_1} \ll 1$
$\deg(g, \mathcal{U}, y) = \deg(f, \mathcal{U}, y)$

For $|y_1 - y_0| \ll 1$, $\deg(f, \mathcal{U}, y) = \deg(f, \mathcal{U}, y_1)$

(P7) EXCISION: If $K \subseteq \bar{\mathcal U}$, closed and $y \notin f(K) \cup f(\partial \mathcal U)$

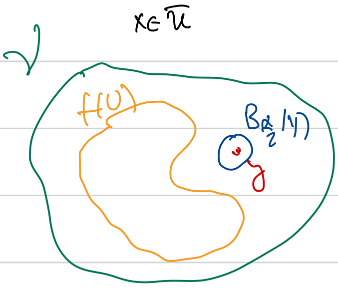
$$\deg(f, \mathcal U, y) = \deg(f, \mathcal U \setminus K, y)$$

All these properties can be proved exploiting properties of n-forms

proof (P1) $\deg(\mathbb{1}_{\mathbb R^n}, \mathcal U, y) = \int_U \mathbb{1}^* \omega = \int_U \omega = \begin{cases} 1 & \text{if } y \in \mathcal U \\ 0 & \text{if } y \notin \mathcal U \end{cases}$

if $y \in \mathcal U$, take $y \in \operatorname{supp}\omega \subset \mathcal U$
$y \notin \mathcal U$ " $y \in \operatorname{supp}\omega \subset \mathcal U^c$

(P2) BC $\inf_{x \in \bar{\mathcal U}} |f(x) - y| \geq \alpha > 0$. Let ω with $y \in \operatorname{supp}\omega \subseteq B_{\frac{\alpha}{2}}(y)$



$$\deg(f, \mathcal U, y) = \int_{\mathcal U} f^* \omega = \int_U \underbrace{\omega(f(x))}_{=0} \det Jf(x)\, dx_1 \wedge \dots dx_n$$

$$= 0 \quad ↯$$

(P3) $\deg(f - p, \mathcal U, 0) = \int_{\mathcal U} (f-p)^* \omega = \int_{\mathcal U} \underbrace{\omega(f(x) - p) \det Jf(x)\, dx_1 \wedge \dots \wedge dx_n}$

ω admissible at 0, $0 \in \operatorname{supp}\omega$

$\omega_p := \omega(\cdot - p)\, dx_1 \wedge \dots \wedge dx_n$, $p \in \operatorname{supp}\omega$

ω_p is admissible at p $\quad = \deg(f, \mathcal U, p)$

(P4) trivial

(P5) Let $\tilde{\mathcal U} := [0,t] \times \mathcal U$. Since $d\omega = 0$ and $h(t,x)$ adm

Choose ω admissible at y and with $\operatorname{supp}\omega \subset \mathcal V \setminus \bigcup_t h(t, \partial \mathcal U)$

$$0 = \int_{[0,t] \times \mathcal U} h^* d\omega = \int_{[0,t] \times \mathcal U} d h^* \omega = \int_{\partial([0,t] \times \mathcal U)} h^* \omega =$$

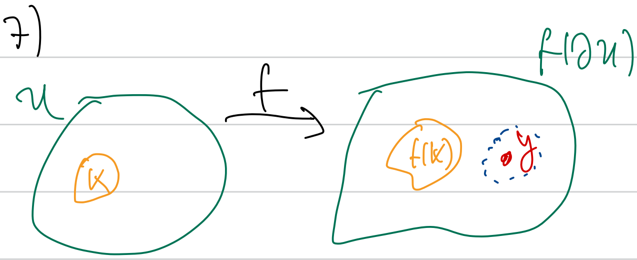
$$= \int_{\{0\} \times \mathcal U} + \int_{\{t\} \times \mathcal U} + \underbrace{\int_{[0,t] \times \partial \mathcal U}}_{=0} = \int_{\mathcal U} h(t,\cdot)^* \omega - \int_{\mathcal U} h(0,\cdot)^* \omega$$

(P6) Put $h(t,x) = (1-t)\, f(x) + t\, g(x)$: it is admissible homotopy provided $\|g - f\|_{L^\infty} \ll \mathrm{dist}(y, f(\partial \mathcal{U}))$

$$\Rightarrow \deg(f, \mathcal{U}, y) = \int f^* \omega = \int h^*(1,\cdot)\omega = \int h^*(0,\cdot)\omega = \int g^* \omega$$
$$= \deg(g, \mathcal{U}, y)$$

Let $|y_1 - y| \ll 1$. Then ω admissible for y is admissible for y_1

(P7) K closed, f continuous $\Rightarrow f(K)$ closed



$\rightsquigarrow \mathrm{dist}(f(K) \cup f(\partial \mathcal{U}), y) \geq \alpha > 0$

take $y \in \mathrm{supp}\, \omega \subset \mathcal{V} \setminus [f(K) \cup f(\partial\mathcal{U})]$

$$\deg(f, \mathcal{U}, y) = \int_{\mathcal{U}} f^* \omega = \int_{\mathcal{U}} \omega(f(x)) \det df(x)\, dx_1 \wedge \dots \wedge dx_n$$
$$= \int_{\mathcal{U} - K} + \underbrace{\int_K}_{=0} = \deg(f, \mathcal{U} \setminus K, y)$$

How to compute the degree?

Def •) $f : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1$, $y \in \mathbb{R}^n$ is a <u>regular value</u> of f if $\forall x \in f^{-1}(y)$, $df(x)$ is invertible

(If $y \notin f(U)$, then y reg value)

•) y is critical value if y not regular value

Thm (Sard) $f \in C^1$, the set of critical values has measure 0

Prop U open bd, $y \notin f(\partial U)$ regular value, then

$\#\{x \in U : f(x) = y\}$ is finite and

if $\{x_1, \dots, x_m\} = f^{-1}(y)$ then $\exists\, U_{x_i}$ neighb of x_i and U_y neigh of y : $f : U_{x_i} \to U_y$ is bijection

and $f^{-1}(U_y) = \bigcup_{i=1}^m U_{x_i}$

proof $f^{-1}(y)$ closed in $\bar{U}$ $\leadsto$ it is compact
By the inverse function thm, f loc diffeom around x $\forall x: f(x)=y$
$\leadsto$ $f^{-1}(y)$ compact set of isolated points
$\leadsto$ it has finite points. $\square$

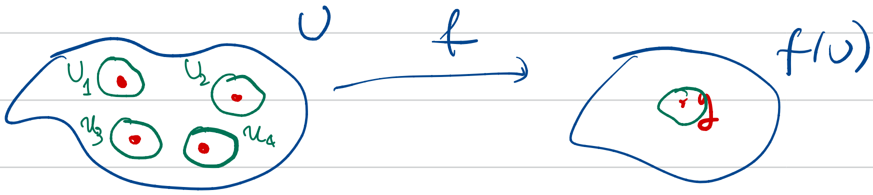
Thm $f \in D_y(U; \mathbb{R}^n)$, $f \in C^1$, y regular value, then

$$\deg(f, U, y) = \sum_{x \in f^{-1}(y) \cap U} \operatorname{sgn}(\det df(x))$$

$$\left(\text{agreement: } \sum_{x \in \emptyset} = 0 \right)$$

proof 1) y reg value, $f^{-1}(y) = \{x_1, \dots, x_n\}$

$\exists\, U_1, \dots, U_n$ open sets st. $y \notin f(\underbrace{U \setminus \bigcup_i U_i}_{K \text{ closed}})$



$$\Rightarrow \deg(f, U, y) = \sum_{i=1}^{n} \deg(f, U_i, y)$$

2) Let $N = \bigcap_{i=1}^{n} f(U_i)$ a neigh of y, ω admissible at y with $\operatorname{supp} \omega \subseteq N$.

$$\deg(f, U_i, y) = \int_{U_i} f^* \omega = \int_{U_i} \omega(f(x)) \det df(x)\, dx_1 \wedge \dots \wedge dx_n$$

$$= \int_{U_i} \omega(f(x))\, |\det J_f(x)|\, \underbrace{\operatorname{sgn}(\det df(x))}_{\substack{\text{constant in } U_i = \\ = \operatorname{sgn} \det df(x_i)}}\, dx_1 \wedge \dots \wedge dx_n$$

$$= \operatorname{sgn}(\det J_f(x_i)) \int_{U_i} \omega(f(x))\, |\det df(x)|\, dx_1 \wedge \dots \wedge dx_n$$

$$= \operatorname{sgn}(\det J_f(x_i)) \underbrace{\int_N \omega}_{=1}$$

$f: U_i \to N$ diffeo, change of variables formula

KEY OBSERVATION 1 $\deg(f, \mathcal{U}, y)$ is always an integer!

Indeed take y_1 close to y regular value ($\exists$ by Sard)

$$\deg(f, \mathcal{U}, y) = \deg(f, \mathcal{U}, y_1) \in \mathbb{N}$$

$\uparrow$ local constancy

KEY OBSERVATION 2 $\deg(f, \mathcal{U}, y)$ extends to continuous functions!

Indeed let $f_n \rightarrow f$ in C^0, $(f_n)_n \subseteq C^1$

Then $\deg(f_n, \mathcal{U}, y) = \deg(f_m, \mathcal{U}, y) \quad \forall n, m \geq N$

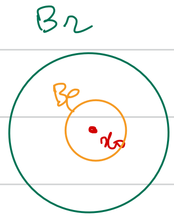
by local constancy. Then we define

$$\deg(f, \mathcal{U}, y) := \lim_{n \to \infty} \deg(f_n, \mathcal{U}, y)$$

KEY OBSERVATION 3 by excision we can define the index of isolated solution of $f(x) = y$.

Indeed assume $\exists$ $r > 0$ s.t. $f(x) \neq y \quad \forall x \in B_r(x_0) \setminus \{x_0\}$

$$\deg(f, B_r(x_0), y) = \deg(f, B_\rho(x_0), y) \quad \forall \rho \in (0, r)$$



$$i(f, x_0) := \lim_{\rho \to 0} \deg(f, B_\rho(x_0), y), \quad y = f(x_0)$$

INDEX of f respect to x_0

EXAMPLE $f \in C^1(\Omega, \mathbb{R}^n) \cap C(\overline{\Omega}, \mathbb{R}^n)$, $y = f(x_0)$ a regular value of f.

$$i(f, x_0) = (-1)^\beta, \quad \beta = \sum \text{ algebraic mult. of negative eigenv. of } f'(x_0)$$

Indeed $i(f, x_0) = \deg(f, \underbrace{B_r(x_0)}_{\text{ball isolating } x_0}, y) = \operatorname{sgn} \det df(x_0)$

$\longrightarrow$ $f \in C^1$ + y reg. value

By Jordan normal form, $\det J_f(x_0) = \lambda_1 \cdots \lambda_n$, λ_j's eigenvalues

·) y reg. value $\Rightarrow$ $\lambda_y \neq 0$ $\forall y$

·) f real valued $\Rightarrow$ if λ_y eigenv., $\bar{\lambda}_y$ eigenvalue

$\rightsquigarrow$ sgn det $J_f(x_0)$ = # negative eigenvalues of $f'(x_0)$

Thm (Brouwer's fixed point thm) Let U open set with $\overline{U}$ homeomorphic to $\overline{B_1(0)} \subset \mathbb{R}^n$ and

$$f: \overline{U} \to \overline{U} \quad \text{continuous}$$

$\Rightarrow$ $\exists\, x \in \overline{U}$: $f(x) = x$

proof Let $\varphi: \overline{U} \to \overline{B_1(0)}$ homeomorphism

then $g = \varphi \circ f \circ \varphi^{-1}: \overline{B_1(0)} \to \overline{B_1(0)}$ continuous

If g fixed point x: $g(x) = x$ $\rightsquigarrow$ $f(\varphi^{-1}(x)) = \varphi^{-1}(x)$

$\rightsquigarrow$ we can assume $U = B_1(0)$

Case 1 $\exists\, x \in \partial B_1(0)$: $f(x) = x$; nothing to do

Case 2 $f(x) \neq x$ $\forall x \in \partial B_1(0)$

strategy: show $\deg(x - f(x), B_1(0), 0) \neq 0 \Rightarrow \exists x: f(x) = x$

To compute degree, put

$$h(t,x) = x - t f(x) \qquad : \qquad \begin{aligned} h(0,x) &= x = \mathbb{I} \\ h(1,x) &= x - f(x) \end{aligned}$$

admissible $\Leftrightarrow$ $h(t,x) \neq 0$ $\forall x \in \partial B_1(0)$ $\forall t \in [0,1]$

otherwise $x = t f(x)$ for some $x \in \partial B_1(0)$, $t \in [0,1)$

$\rightsquigarrow$ $1 = |x| = t |f(x)| \leq t$ $\rightsquigarrow$ $t = 1$

$\rightsquigarrow$ $x = f(x)$ for some $x \in \partial B_1(0)$ ↯

$$\Rightarrow \deg(x - f(x), B_1(0), 0) = \deg(\underbrace{x}_{\mathbb{I}}, B_1(0), 0) = 1$$

☒

LERAY-SCHAUDER DEGREE

look for a degree for $F: \mathcal{U} \subseteq X \to X$, X Banach

EXAMPLE: $F: B_1^{\ell^2(\mathbb{N})}(0) \subseteq \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$

$$\underline{x} = (x_1, x_2, \dots) \longmapsto \left(\sqrt{1-\|x\|^2}, x_1, x_2, \dots\right)$$

F continuous, $F(B_1(0)) \subseteq \partial B_1(0) = \{x \in \ell^2 : \|x\| = 1\}$

one of consequence of degree theory was Brouwer fixed point

$F: \overline{B_1} \to \overline{B_1}$, assume $\exists\ x \in \overline{B_1} : x = F(x)$

$\rightsquigarrow \|x\| = \|F(x)\| = 1 \rightsquigarrow x = F(x)$

$$\Updownarrow$$

$x_1 = 0$

$x = 0$ ↯ $\Longleftarrow$ $x_2 = x_1 = 0$

$x_j = 0 \ \forall j$

$\rightsquigarrow$ Brouwer thm fails in ∞-dim spaces!

Need extra assumption: $F \sim$ compact perturb of $\mathbb{1}$

Def $F: \mathcal{U} \subseteq X \to X$, X Banach, F continuous is said to be <u>compact</u> iff.

$\forall\ B \subseteq \mathcal{U}$ bd, $\overline{F(B)}$ is compact

<u>Rem</u> •) F is nonlinear function. If $F \in \mathcal{L}(X)$ and compact according to def $\rightsquigarrow$ old notion of compactness

a) $X = C^0([0,1])$; $f : \mathbb{R} \to \mathbb{R}$ continuous

$$F : X \to X \qquad u \mapsto F(u)(t) = \int_0^t f(u(s))\, ds$$

F is compact (Ascoli - Arzela)

Prop Let $(F_j)_{j \geq 1}$, $F_j : \mathcal{U} \to X$ compact $\forall j$ and such that $F_j \to F$ in the sup norm to some $F : \mathcal{U} \to X$ continuous.
Then F is compact.

proof Let $B \subseteq U$ bounded.

claim $\forall \varepsilon > 0$ $\exists N_\varepsilon \in \mathbb{N}$: $\overline{F(B)} \subseteq \bigcup_{i=1}^{N_\varepsilon} B_\varepsilon(y_i)$

This is true for $\overline{F_j(B)}$ since it is totally bounded.
Now given $\varepsilon > 0$, take j : $\|F - F_j\|_\infty \leq \frac{\varepsilon}{2}$

As $\overline{F_j(B)}$ compact $\leadsto \exists$ $\frac{\varepsilon}{2}$ - net for $\overline{F_j(B)}$:

$$\overline{F_j(B)} \subseteq \bigcup_{i=1}^{N(\varepsilon)} B_{\frac{\varepsilon}{2}}(y_i)$$

This is ε - net for $\overline{F(B)}$: $\forall y \in F(B)$, $\exists y_a$ $a = 1, \dots, N_\varepsilon)$ with $\|y - y_a\| < \varepsilon$, indeed

$$\|y - y_a\| \overset{y = F(x),\, x \in B}{=} \|F(x) - y_a\| \leq \|F(x) - F_j(x)\| + \|F_j(x) - y_a\|$$

$$\leq \|F - F_j\|_\infty + \|F_j(x) - y_a\|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$ take the y_a with $F_j(x) \in B_{\frac{\varepsilon}{2}}(y_a)$

$$\rightsquigarrow F(B) \subseteq \bigcup_{i=1}^{N} B_\varepsilon(y_i) \rightsquigarrow \overline{F(B)} \subseteq \bigcup_{i=1}^{N} B_\varepsilon(y_i)$$

Cor $F: X \to X$ continuous and $\sup_{x \in X} \|F(x)\| < \infty$ and $\exists (F_j)_j$ s.t. $F_j \to F$ in the sup norm and $\forall j$: $\dim(\operatorname{Im} F_j(B)) < \infty$ $\forall B$ bd

$\Rightarrow F$ compact

Rem For linear maps:
- for Hilbert spaces, also converse was true.
- for Banach space, it was false

Dropping linearity, also converse is valid

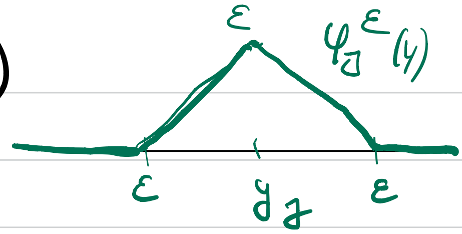
Prop $F: \mathcal{U} \subseteq X \to X$, $\mathcal{U}$ open bounded, and F compact. Then $\forall \varepsilon > 0$ $\exists F_\varepsilon$ continuous such that

$$\|F - F_\varepsilon\| \leq \varepsilon \quad \& \quad \dim(\operatorname{Im} F_\varepsilon) < \infty$$

proof Let $\varepsilon > 0$, $y_1, \dots, y_p$: $\overline{F(\mathcal{U})} \subset \bigcup_{i=1}^{p} B_\varepsilon(y_i)$

and we can choose $(y_i)_{i=1,\dots,p} \in \overline{F(\mathcal{U})}$

Let $\varphi_J^\varepsilon(y) := \max(0, \varepsilon - \|y - y_J\|)$



If $y \in \overline{F(U)}$, $\exists\, J$: $\varphi_J^\varepsilon(y) \neq 0$

$\rightsquigarrow$ $\psi_J^\varepsilon(y) = \dfrac{\varphi_J^\varepsilon(y)}{\sum_a \varphi_a^\varepsilon(y)}$ is well defined for $y \in \overline{F(U)}$

and $\sum_J \psi_J^\varepsilon(y) = 1$

Put $F_\varepsilon(x) = \sum_{a=1}^{p} \psi_a^\varepsilon(F(x))\, y_a$ $\quad (*)$

•) F_ε continuous function

•) $\operatorname{Im} F_\varepsilon \subseteq \operatorname{span}(y_1, \dots, y_p)$ $\rightsquigarrow$ F_ε compact

•) $\|F(x) - F_\varepsilon(x)\| = \Big\| \underbrace{\sum_{a=1}^{p} \psi_a^\varepsilon(F(x))}_{=1} F(x) - F_\varepsilon(x) \Big\|$

$\overset{(*)}{\leq} \sum_{a=1}^{p} \underbrace{\psi_a^\varepsilon(F(x))}_{\neq 0 \text{ only if } \|F(x) - y_a\| < \varepsilon} \overbrace{\|F(x) - y_a\|}^{<\varepsilon} \leq \varepsilon$

$\sum_{a=1}^{p} \psi_a^\varepsilon(y) = 1$

$\rightsquigarrow$ $\|F - F_\varepsilon\|_\infty \leq \varepsilon$ $\qquad \boxed{\text{m}}$

Lemma $F: \mathcal{U} \subseteq X \to X$, $\mathcal{U}$ open bd and F compact, then $\mathbb{1} + F$ is closed (it maps a closed set into closed sets)

<u>proof</u> $B \subseteq \mathcal{U}$ closed, $(x_n)_n \subseteq B$: $x_n + F(x_n) \to y$

we want to show $y = x + F(x)$.

Since F compact, $\exists\ (x_{n_k})$ st. $F(x_{n_k}) \to \bar{y}$

so $$x_{n_k} = \underbrace{x_{n_k} + F(x_{n_k})}_{\downarrow\, y} - \underbrace{F(x_{n_k})}_{\downarrow\, \bar{y}} \to y - \bar{y}$$

$\leadsto \underbrace{y - \bar{y}}_{\shortparallel\, x} \in B$ since B is closed

By continuity of F: $x_{n_k} + F(x_{n_k}) \to x + F(x)$
$\searrow y$

$\blacksquare$

Leray - Schauder degree

take $G = \mathbb{1} + F$, F compact

We want to define a degree for $\mathbb{1} + F$ satisfying (D1) - (D4) as in the finite dimensional case

<u>Idea</u>: try with finite dimensional approximations

Use prop to approximate F with finite dim range map F_ε acting on finite dim space X_ε : $(\mathbb{1} + F_\varepsilon)|_{X_\varepsilon}$

We shall define the degree as a limit of

$\deg\left((\mathbb{1} + F_\varepsilon)_{|X_\varepsilon} \right) \leadsto$ Brouwer degree of this map

To prove that such a limit is well defined, we need an additional property of Brouwer degree

Idea: $\mathcal{U} \subseteq \mathbb{R}^n$: $f: \overline{\mathcal{U}} \to \mathbb{R}^m$, $m < n$, $y \in \mathbb{R}^m \setminus f(\partial \mathcal{U})$. Put

$$g = \mathbb{1}_{\mathbb{R}^n} + \begin{pmatrix} f \\ 0 \end{pmatrix} \begin{matrix} \} \ m \text{ components} \\ \} \ n-m \text{ components} \end{matrix}$$

Take $y \in \mathbb{R}^m \sim \mathbb{R}^m \times \underbrace{\{0\}}_{n-m \text{ times}} \subseteq \mathbb{R}^n \rightsquigarrow y = \begin{pmatrix} y \\ 0 \end{pmatrix}$

If $x \in \mathbb{R}^n$: $g(x) = \begin{pmatrix} y \\ 0 \end{pmatrix} \iff x + \begin{pmatrix} f(x) \\ 0 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$

$\Downarrow$

$x \in \mathbb{R}^m$

$\rightsquigarrow \deg(\mathbb{1} + f, \mathcal{U}, y)$ should be the same as $\deg(\mathbb{1} + f|_{\mathcal{U} \cap \mathbb{R}^m}, \mathcal{U} \cap \mathbb{R}^m, y)$

Lemma (Reduction property of Brouwer degree)

Let $f: \overline{\mathcal{U}} \subseteq \mathbb{R}^n \to \mathbb{R}^m$, continuous, $m < n$, $\mathcal{U}$ open, bd, $y \in \mathbb{R}^m \setminus (\mathbb{1} + f)(\partial U)$. Then

$$\deg(\mathbb{1} + f, U, y) = \deg((\mathbb{1} + f)|_{\mathcal{U}_m}, \mathcal{U}_m, y)$$

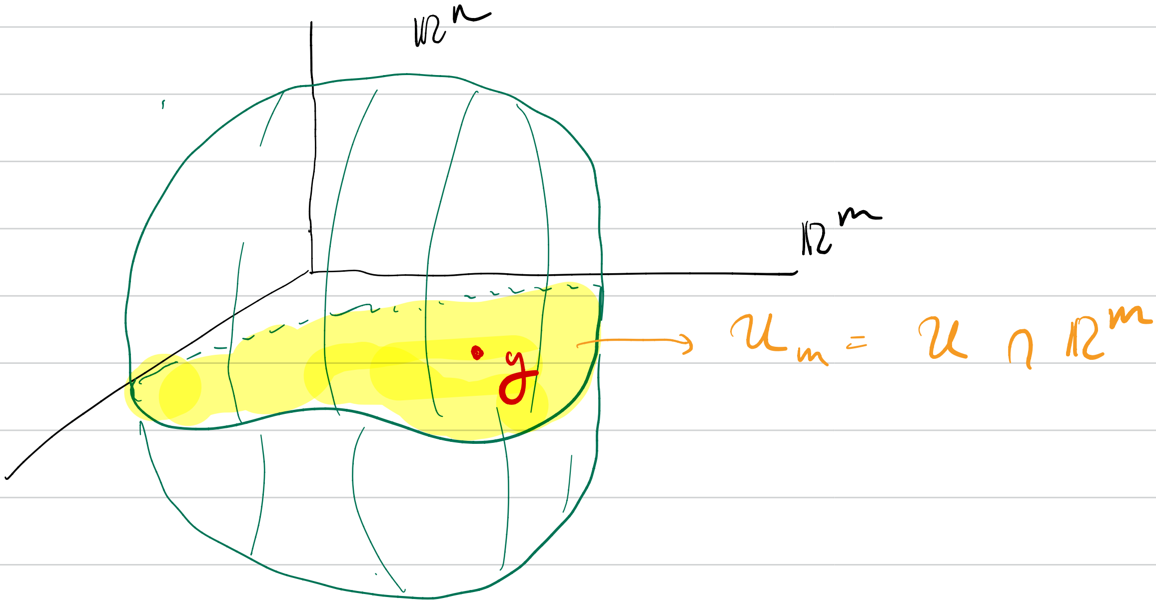
where $\mathcal{U}_m = \mathcal{U} \cap \mathbb{R}^m \equiv \mathcal{U} \cap (\mathbb{R}^m \times \{0\})$

Rem •) $\mathbb{R}^m \simeq \mathbb{R}^m \times \underbrace{\{0\}}_{n-m \text{ components}} \subseteq \mathbb{R}^n.$

$$f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$
$$x \longmapsto \begin{pmatrix} f(x) \\ 0 \end{pmatrix} \begin{matrix} \}\, m\text{-components} \\ \}\, n-m \text{ components} \end{matrix}$$

•) $\begin{pmatrix} \overset{y}{\overset{\shortparallel}{y}} \\ 0 \end{pmatrix} \in \left(\mathbb{1} + \begin{pmatrix} f \\ 0 \end{pmatrix} \right) (\partial U)$

•)



•) let $x \in \overline{U}$ s.t. $x + f(x) = y \Rightarrow x = \underset{\mathbb{R}^m}{\underset{\cap}{y}} - \underset{\mathbb{R}^m}{\underset{\cap}{f(x)}}$

$\rightsquigarrow x \in \mathbb{R}^m$

$\rightsquigarrow (\mathbb{1} + f)(x) = (\mathbb{1} + f)_{|\mathcal{U}_m}(x) = y$

$\rightsquigarrow x \in (\mathbb{1} + f)_{|\mathcal{U}_m}^{-1}(y)$

$\Rightarrow (\mathbb{1} + f)^{-1}(y) = \left((\mathbb{1} + f)_{|\mathcal{U}_m} \right)^{-1} (y)$

proof By usual approximation, we can assume $f \in C^1$ and y is regular value. It is enough to prove

$$\operatorname{sign} \det d(\mathbb{1}+f)(x) = \operatorname{sign} \det d\left((\mathbb{1}+f)_{|\mathcal{U}_m}(x)\right)$$

$$\forall\, x \in (\mathbb{1}+f)^{-1}(y) = \left((\mathbb{1}+f)_{|\mathcal{U}_m}\right)^{-1}(y)$$

Recall $f(x) = \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$ $\begin{matrix} m \text{ comp} \\ n-m \text{ comp.} \end{matrix}$

$\rightsquigarrow \det\left(d(\mathbb{1}+f)(x)\right) =$

$$= \det \begin{array}{c} \begin{bmatrix} \mathbb{1}_m + \partial_j f_i & \partial_j f_i \\ 0 & \mathbb{1}_{n-m} \end{bmatrix} \begin{matrix} \}\, m \\ \}\, n-m \end{matrix} \\ \underbrace{\qquad\qquad\qquad}_{m} \quad \underbrace{\qquad\quad}_{n-m} \end{array}$$

develop w.r.t. last $n-m$ rows

$$= \det d(\mathbb{1}_m + f)(x)$$

$\rightsquigarrow$ the 2 degrees coincide ☒

Define

$$K(\mathcal{U}, X) = \{ F \in C^0(\mathcal{U}, X),\ F \text{ compact} \}$$

$$F(\mathcal{U}, X) = \{ F \in C^0(\mathcal{U}, X),\ \operatorname{Im} F \text{ finite dim} \}$$

$$D_y(\mathcal{U},X) = \{ F \in K(\overline{\mathcal{U}},X) ; \quad y \notin (\mathbb{1}+F)(\partial U) \}$$

$$\mathcal{F}_y(\mathcal{U},X) = \{ F \in \mathcal{F}(\overline{\mathcal{U}},X) : \quad y \notin (\mathbb{1}+F)(\partial U) \}$$

Note that if $F \in D_y(\mathcal{U},X)$, then

$$\mathrm{dist}\,(y, \underbrace{(\mathbb{1}+F)(\partial U)}_{\text{closed set}}) > 0$$

(exercise)

Put
$$\rho := \mathrm{dist}\,(y, (\mathbb{1}+F)(\partial U)) > 0$$

then approximate F with $F_1 \in \mathcal{F}(\mathcal{U}, X)$ so that

$$\| F - F_1 \|_\infty < \frac{\rho}{2}$$

$\rightsquigarrow$ $\mathrm{dist}(y, (\mathbb{1} + F_1)(\partial U)) > 0$ $\rightsquigarrow$ $F_1 \in \mathcal{F}_y(\mathcal{U},X)$

Next, take $X_1 \subset X$ finite dim subspace of X

with
$$\begin{cases} F_1(\mathcal{U}) \subset X_1 \\ y \in X_1 \end{cases}$$

and set $\mathcal{U}_1 := \mathcal{U} \cap X_1$, then we

have also $F_1 \in \mathcal{F}_y(\mathcal{U}_1, X_1)$

We put

$$\left[\ \deg(\mathbb{1}+F, \mathcal{U}, y) := \deg(\mathbb{1}+F_1, \mathcal{U}_1, y)\ \right]$$

LERAY - SCHAUDER DEGREE

Prop The Leray - Schauder degree is well posed.

proof Pick $F_2 \in \mathcal{F}(\overline{\mathcal{U}}, X)$: $\|F_2 - F\|_\infty < \rho/2$

X_2 as before

and define $X_0 := X_1 + X_2$

$U_0 = \mathcal{U} \cap X_0$

By reduction thm ($y \in X_1$, $y \in X_2$)

$$\deg(\mathbb{1}+F_1, \mathcal{U}_0, y) = \deg(\mathbb{1}+F_1, \mathcal{U}_1, y)$$

$$\deg(\mathbb{1}+F_2, \mathcal{U}_0, y) = \deg(\mathbb{1}+F_2, \mathcal{U}_2, y)$$

Put $H(t) = \mathbb{1} + (1-t)F_1 + tF_2$

It is admissible homotopy since

$$\|H(t) - (\mathbb{1}+F)\|_\infty \leq \|F_1 - F\|_\infty + \|F_2 - F\|_\infty < \rho$$

$$\rightsquigarrow \deg(\mathbb{1}+F_1, \mathcal{U}_1, y) = \deg(\mathbb{1}+F_1, \mathcal{U}_0, y)$$

homotopy invariance of Brouwer degree

$$= \deg(\mathbb{1}+F_2, \mathcal{U}_0, y)$$

$$= \deg(\mathbb{1}+F_2, \mathcal{U}_2, y)$$

Thm $\mathcal{U}$ open bd, $\mathcal{U} \subseteq X$, $F \in D_y(\mathcal{U}, X)$, $y \in X$
then Leray – Schauder degree fulfills (P1) – (P4)
and (P5) iff homotopy $h(t,\cdot)$ compact perturbation of identity $\forall t$.

proof exercise

Cor The additional properties of Brouwer degree deriving from (D1) – (D4) holds true for Leray-Schauder degree.

Application : Schauder fixed point thm

Thm Let D a closed convex bd subset of X Banach and
$$F : D \to D \quad \text{compact}$$
Then F has a fixed point (i.e. $\exists\, x \in D$: $F(x) = x$)

proof Assume that $0 \in D$ (otherwise translate the set)

Case 1 If $\exists\, x \in \partial D$ with $F(x) = x$ ✓

Case 2 $\forall x \in \partial D$: $F(x) \neq x$

So we can define $\deg(\mathbb{1} - F, D, 0)$ and show to

Put $\quad h(t,x) = x - tF(x), \quad t \in [0,1], \; x \in D$

$\forall t \in [0,1]$, $\quad h(t,\cdot) = \mathbb{1} + \text{compact}$

Let us show h is admissible: we claim that

$$h(t,x) \neq 0 \quad \forall \; (t,x) \in [0,1] \times \partial D$$

otherwise: $\quad \exists \; (\bar{t}, \bar{x}) \in [0,1] \times \partial D$ with $h(\bar{t},\bar{x}) = 0$, i.e.

$$\bar{x} = \bar{t} \, F(\bar{x})$$

- $F(x) \neq x \quad \forall x \in \partial D \quad \Rightarrow \quad \bar{t} < 1$
- $F(D) \subseteq D \quad \Rightarrow \quad F(\bar{x}) \in D$
- D convex $\quad \Rightarrow \quad \bar{t} \, F(\bar{x}) \in D$
- $\bar{t} < 1 \quad \Rightarrow \quad \bar{t} \, F(\bar{x}) \in \mathring{D} \quad \lightning$

$\partial D \ni \bar{x} = \bar{t} \, F(\bar{x})$

$\rightsquigarrow \quad h(t,\cdot)$ admissible

$$\Rightarrow \quad \underbrace{\deg(h(0,\cdot), D, 0)}_{\substack{\| \\ \deg(\mathbb{1}, D, 0) \\ \| \\ 1}} = \underbrace{\deg(h(1,\cdot), D, 0)}_{\substack{\| \\ \deg(\mathbb{1} - F, D, 0)}}$$

(D2) $\quad 0 \in D$

$\Rightarrow \quad \deg(\mathbb{1} - F, D, 0) \neq 0 \quad \Rightarrow \quad \exists$ sol of $(\mathbb{1} - T)(x) = 0$

$\square$

Peano theorem

Let $f \in C(I \times \mathcal{U}, \mathbb{R}^n)$ with $I \subseteq \mathbb{R}$, $\mathcal{U} \subseteq \mathbb{R}^n$ open and consider

$$(P) \begin{cases} \dot{x} = f(t, x(t)) \\ x(t_0) = x_0 \end{cases} \quad \text{with } t_0 \in I, \ x_0 \in \mathcal{U}.$$

Then (P) has at least 1 solution $x(t) \in C^1((t_0-\varepsilon, t_0+\varepsilon), \mathbb{R}^n)$

proof integral formulation: $x(t)$ is a sol iff

$$x(t) = x_0 + \int_{t_0}^{t} f(t, x(t)) = F(x(t))$$

the map $F: C([0,1]) \longrightarrow C([0,1])$ is compact.

CHECK IT: (Ascoli–Arzelà)

Let $M := \sup_{t \in I, \ \|x - x_0\| \leq \rho} \|f(t,x)\|$

and consider $D := \{ y(t) \in C^0([t_0-\delta, t_0+\delta], \mathbb{R}^n) : \sup_t \|y(t) - x_0\| \leq \rho \}$

Then •) D closed and convex (it is a ball)

•) Does $F(D) \subseteq D$? Let $x(t) \in D$, then

$$\|F(x(t)) - x_0\| = \left\| \int_{t_0}^{t} f(t, x(t))\, dt \right\|$$

$$\leq \delta M \leq \rho$$

provided δ is chosen suff. small!

$\rightsquigarrow$ by Schauder's theorem $\exists$ fixed point of F
it is a solution of (P)

Sturm-Liouville problem

$$(D) \begin{cases} -u'' = f(x, u(x)) \\ u(0) = u(1) = 0 \end{cases}$$

Thm Let $f: \Omega \times \mathbb{R} \to \mathbb{R}$ continuous and bounded. Then (D) has a classical solution.

proof Consider the auxiliary linear problem

$$\begin{cases} -u'' = g \\ u(0) = u(1) = 0 \end{cases}$$

By Sturm-Liouville theory, the sol op

$$T: L^2 \to L^2 \quad \text{is compact}$$
$$g \mapsto Tg$$

Consider $F(u) := T[f(x, u(x))]$

-) F continuous: f is C^0 and bd, $u \mapsto f(x,u)$ is a Nemytzkii operator, continuous $L^2 \to L^2$
-) F compact: ok because T is compact
-) $F: D \to D$ with D closed and convex;

Put $M := \sup_{x \in \Omega,\, t \in \mathbb{R}} |f(x,t)|$

$$D := \{ u \in L^2 : \|u\|_{L^2} \leq R \}$$

Then $\| F(u) \|_{L^2} \le C_1 \| f(x, u(x)) \|_{L^2} \le C_1 M \le R$
provided R large enough

$\Longrightarrow$ $\exists$ u fixed point in L^2 of $u = F(u)$
i.e. $u = T[f(x, u(x))]$

$\rightsquigarrow$ $f: L^2 \to L^2$, $T: L^2 \to H^1_0$ so $u \in H^1_0$

so u is weak sol $\Rightarrow$ $u(0) = u(1) = 0$.
Then $u \in H^1_0 \hookrightarrow C([0,1])$ and so $-u'' = f(x, u(x)) \in C^0$
$\Rightarrow$ $u \in C^2([0,1])$ and $u(0) = u(1) = 0$ $\to$ classical solution.

How to compute the degree of a map?

Thm Let $T \in C^1(\overline{U}, X)$ be a compact map.
X reflexive
Put $S(x) = x - T(x)$
and let x_0 be isolated solution $S(x_0) = y$.
If $1 \notin \sigma(T'(x_0))$ then

$$i(S, x_0) := \deg(S, B_\varepsilon(x_0), y) = (-1)^\beta$$

with $\beta = \sum$ algebraic multiplicity of eigenvalues of $T'(x_0) > 1$

Rk If $T \in C^1$ is compact, then $T'(x_0)$ is compact (exercise). Then # eigenvalues >1 is finite
so index is well defined

proof Step 1 Reduction to linear part

Wlog $x_0 = 0$, $y = 0$. By Taylor

$$S(x) = \underbrace{x - T'(0)[x]}_{S'(0)} + R(x), \qquad R(x) = o(\|x\|)$$

Define the homotopy $h(t,x) = x - T'(0)[x] + tR(x)$

$h(t,\cdot)$ compact perturbation of identity $\forall t \in [0,1]$

CLAIM $\exists r>0$ suff. small so that $h(t,x)$ is admissible at 0 on $B_r(0)$

i.e. $h(t,x) \neq 0 \quad \forall (t,x) \in [0,1] \times \partial B_r(0)$

BC: $\forall n, \ \exists (t_n, x_n) \in [0,1] \times \partial B_{\frac{1}{n}}(0)$ st $h(t_n, x_n) = 0$

put $z_n = \dfrac{x_n}{\|x_n\|}$: $\quad z_n = T'(0)[z_n] - t_n \dfrac{R(x_n)}{\|x_n\|}$

then (z_n) bd, $z_n \rightharpoonup z^*$, $t_n \to t^*$

Both $T'(0)$ and $R(x)$ are compact $\Rightarrow$

$$T'(0)[z_n] - t_n \frac{R(x_n)}{\|x_n\|} \longrightarrow T'(0) z$$

$\Rightarrow z = T'(0) z$ ↯ $1 \notin \sigma(T'(0))$

$\rightsquigarrow z_n \to z$ strongly and $\|z\| = 1$

$$\Rightarrow \deg(S, U, 0) = \deg(\mathbb{1} - T'(0), U, 0)$$

Step 2 If T compact op, $1 \notin \sigma(T)$, compute $\deg(\mathbb{1} - T, B_r(0), 0)$

Split $X = N \oplus W$ with
- N, W invariant
- $\sigma(T|_N) = \{\lambda \in \sigma(T) : |\lambda| > 1\}$
- $\sigma(T|_W) = \sigma(T) - \sigma(T|_N)$

Let P_N, P_W project on N & W

Let $h(t,x) = x - TP_N x - t\, TP_W x$ homotopy

- $h(t,\cdot) = \mathbb{1} + \text{compact} \quad \forall t$

i) $h(t,x)$ admissible : $\underline{BC}$ $\exists$ (t_*, x_*) s.t. $\begin{cases} h(t_*, x_*) = 0 \\ \|x_*\| = 1 \end{cases}$

$$x_* = T P_N x_* + t_* T P_W x_*$$

Apply P_N: $\quad P_N x_* - \underbrace{P_N T P_N}_{P_N \text{ leaves } T \text{ invariant}} x_* = t_* \underbrace{P_N T P_W x_*}_{=0}$

$\Rightarrow \quad P_N x_* = T P_N x_*$, but $1 \notin \sigma(T)$

$\Rightarrow P_N x_* = 0 \quad \Rightarrow \quad x_* = P_W x_*$

Apply P_W: $\quad P_W x_* = t_* T P_W x_*$

then if $t_* = 0 \Rightarrow P_W x_* = 0 \Rightarrow x_* = 0$ ↯

if $t_* = 1 \Rightarrow 1 \in \sigma(T)$ ↯

$\rightsquigarrow t_* \in (0,1) \rightsquigarrow \frac{1}{t_*} \in (1, +\infty)$ eigenvalue of T

but $\sigma(T_{|W}) = \sigma(T) \setminus \sigma(T_{|N}) = \{\lambda \in \sigma(T) : |\lambda| < 1\}$ ↯

By invariance property of degree

$$\deg(\mathbb{1} - T, B_2(0), 0) = \deg(\mathbb{1} - T_{|N}, B_2(0), 0)$$

$$= (-1)^\beta \quad , \quad \beta = \# \text{ eigenv. of } \mathbb{1} - T_{|N} < 0$$

But μ eig. of $\mathbb{1} - T \iff \lambda = 1 - \mu$ eig of T

$\rightsquigarrow \#\{\mu < 0 \text{ eig of } \mathbb{1} - T_{|N}\} = \#\{\lambda > 1 \text{ eig of } T\}$

$\square$